Pergamon

J. Appl. Maths Mechs, Vol. 59, No. 5, pp. 769–772, 1995

0021–8928/95 \$24.00+0.00

0021–8928(95)00090-9

DAMPING OF FREE OSCILLATIONS IN MOVING ELASTIC SYSTEMS†

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(*Received* 22 *June* 1994)

The characteristic equation for a linear elastic system having zero natural frequencies to which a linear elasto-viscous oscillation damper is attached is derived. It is shown that if this damper has one degree of freedom and is not connected to a fixed base, the characteristic equation can be reduced to a form similar to that obtained for the case when the elastic system has no zero natural frequencies. The problem of finding the values of the damper parameters for which the minimum of all the attenuation decrements, corresponding to the least non-zero natural frequencies, has the maximum possible value, is similar to the same problem for a non-free elastic system.

Many papers have been published on the subject of damping of oscillations in elastic systems using elasto-viscous dampers (for example, [1–3]). The problem of finding those damper parameters for which the minimum of all the damping decrements in a two-mass system has the maximum possible value was solved in [4]. This problem was extended in [5] to the case when a damper with one degree of freedom is connected to an elastic system with an arbitrary number of degrees of freedom, but only decrements corresponding to the least natural frequencies were considered. A characteristic equation was derived in [6] for an elastic system with an elasto-viscous damper of a fairly general form connected to it.

The results obtained relate to the case when the elastic system is fixed and dynamically stable, when all its natural frequencies are real and positive. However, as will be shown below, many of these results can also be extended to the case when the elastic system has a finite number of zero natural frequencies.

Suppose a free elastic system of finite mass M occupies a volume V and has N oscillatory degrees of freedom and N_1 "non-oscillatory" degrees of freedom (corresponding to its displacement as a rigid solid). Of these degrees of freedom there are not more than three translational and three rotational, and hence $N_1 \leqslant 6$. If $\rho(X)$ is the matrix of generalized densities of the elastic system, and $\Omega_1 \leqslant \Omega_2 \leqslant \ldots \leqslant \Omega_N$ are its non-zero natural frequencies, all the eigenfunctions of the elastic system can be orthonormalized to its mass

$$\int_V \mathbf{u}'_{0j}(X)\rho(X)\mathbf{u}_{0k}(X)dV(X) = M\delta_{jk} \quad (1 \leqslant j, k \leqslant N_1)$$

$$\int_V \mathbf{u}'_{0j}(X)\rho(X)\mathbf{u}_{k}(X)dV(X) = 0 \quad (1 \leqslant j \leqslant N_1,\ 1 \leqslant k \leqslant N)$$

$$\int_V \mathbf{u}'_{j}(X)\rho(X)\mathbf{u}_{k}(X)dV(X) = M\delta_{jk} \quad (1 \leqslant j,\ k \leqslant N)$$

Here $\mathbf{u}_{0j}(X)$ $(1 \leqslant j \leqslant N_1)$ are the eigenfunctions corresponding to the zero natural frequencies, $\mathbf{u}_j(X)$ $(1 \leqslant j \leqslant N)$ are the corresponding non-zero natural frequencies and $dV(X)$ is an element of volume at the point X. Generally speaking it is possible to have $N = \infty$.

If a linear elasto-viscous damper, the transfer matrix of which is known, is attached to the elastic system, a characteristic equation can be derived for this system with the damper in the same way as when there are no zero natural frequencies. However, zero roots can also occur; they correspond to the combined motion of the elastic system and the damper. The problems involved in optimizing the damper parameters in this case can be formulated and solved in the same way as for fixed elastic systems, but the zero roots are ignored. If the damper has the form of a point mass connected to the elastic system with a linear elasto-viscous coupling, it is necessary to obtain those values of the parameters of the coupling (the stiffness and viscosity), that will maximize the minimum of the damping decrements, corresponding to the least non-zero natural frequencies [6]. This problem can be inverted and extended

†*Prikl. Mat. Mekh.* Vol. 59, No. 5, pp. 799–802, 1995.

to a more complex case, namely, to obtain, for a damper of given construction, those values of the mass and parameters of the couplings so that the overall mass of the damper is the minimum possible and all the attenuation decrements in a specified frequency range are not less than a specified value, depending on it [7].

We will show that if a free point damper with one degree of freedom is connected to an elastic system, the problem of optimizing the parameters of this damper can be reduced to the analogous problem for a fixed elastic system. Suppose the damper has a mass m, is connected to a point L by a spring of stiffness c and with a damper having a coefficient of viscous friction h, and can be displaced in the direction of the unit vector n (Fig. 1). Then, using the dimensionless parameters

$$\theta=\frac{m}{M},\quad \sigma=\frac{c}{M\Omega_1^2},\quad z=\frac{h}{M\Omega_1},\quad \upsilon_0=\sum_{j=1}^{N_1}[\mathbf{n}'\mathbf{u}_{0j}(L)]^2,\quad \upsilon_j=\mathbf{n}'\mathbf{u}_j(L),$$

$$\omega_j=\frac{\Omega_j}{\Omega_1}\ (1\leqslant j\leqslant N),\quad r=-\frac{\lambda}{\Omega_1} \tag{1}$$

(where λ is a characteristic index) the characteristic equation can be written in the form

$$r^{2N_1}\left\{\prod_{j=1}^{N}\left[1+\left(\frac{r}{\omega_j}\right)^2\right]\right\}\left[r^2-z'r+\sigma'+\theta' r^2(-z'r+\sigma')\sum_{j=1}^{N}\frac{\upsilon_j^2}{r^2+\omega_j^2}\right]=0 \tag{2}$$

($\theta' = \theta(1+\upsilon_0\theta)^{-1}$, $\sigma' = (1+\upsilon_0\theta)\sigma$, $z' = (1+\upsilon_0\theta)z$), which is analogous to the equation derived previously in [5].

The problem of optimizing the parameters of the damper can be formulated as follows: it is required to obtain, as a function of θ', those values of σ' and z' such that the least of the real parts of the non-zero roots of Eq. (2), the imaginary parts of which are a minimum, are the maximum possible. If we are dealing with the damping of the first mode of free oscillations, and $\theta' \ll 1$, then to achieve this maximum, Eq. (2) should have a pair of double complex-conjugate roots close to $\pm i$; the real parts of these roots are equal to $n = |\upsilon_1|\sqrt{(\theta')}[1+O(\theta')]/2$. As θ' increases the necessary conditions for σ' and z' to be optimum in this sense may change. As $\theta \to \infty$ the parameter θ' has a finite limit υ_0^{-1}; r, n, σ', z' and the dimensional parameters of the couplings $c = M\Omega_1^2\sigma'\upsilon_0^{-1}$ and the dimensional parameters of the couplings $h = M\Omega_1 z'\upsilon_0^{-1}$ also have finite limits.

If a more complex damper is connected to the elastic system, the problem cannot be reduced to the case when $N_1 = 0$. If the damper consists of a set of several point masses connected by elasto-viscous couplings to the same point of the elastic system and which is displaced in the same direction, and it is necessary to obtain those values of the damper parameters such that its overall mass is a minimum, and the damping decrement of the first few natural frequencies is not less than a given value, the problem can be solved in the same way as the case of a fixed elastic system. For small values of the damping decrement n the characteristic equation must have double complex roots with a common real part n. The values of the masses which form the damper are proportional to n^2; in particular, for the case considered above θ' and θ are equal to $4(\upsilon_1^{-1}n)^2[1+O(n^2)]$ (7).

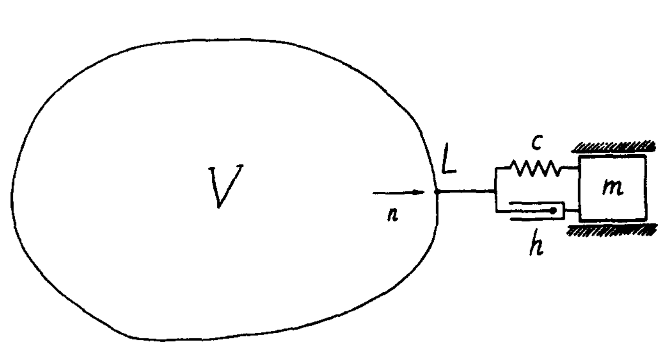

Fig. 1.

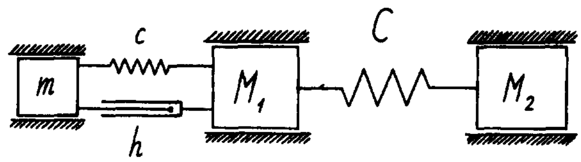

Fig. 2.

As an example we will consider an elastic system with $N_1 = N = 1$. It may consist of two point masses M_1 and M_2 connected by a spring of stiffness C; a point damper of mass m is connected to M_1; it as well as the masses M_1 and M_2 can be displaced in the same direction (Fig. 2). In (1) we have ($M = M_1 + M_2$, $\Omega_1 = [(M_1 + M_2)C(M_1M_2)^{-1}]^{1/2}$, while $\upsilon_0 = 1$, $\upsilon_1 = (M_2M_1^{-1})^{1/2}$. If we introduce the dimensionless parameters

$$\theta'' = \upsilon_1^2\theta' = \frac{mM_2}{M_1(M+m)}, \quad \sigma' = \frac{M_1M_2(M+m)c}{M^2mC}, \quad z' = (M+m)[M_1M_2(M^3C)^{-1}]^{1/2}m^{-1}h$$

Eq. (2) will have the form

$$r^2\{r^4 - (1+\theta'')z'r^3 + [1+(1+\theta'')\sigma']r^2 - z'r + \sigma'\} = 0 \tag{3}$$

and, apart from the factor r^2 on the left-hand side, it is similar to the characteristic equation for a fixed elastic system with one degree of freedom. The dimensionless parameters introduced for this system [4] are identical with the parameters of Eq. (3) as $M_2 \to \infty$. If we consider a fixed elastic system, connected by an elasto-viscous coupling to a fixed foundation [8], the parameters of its characteristic equation will be identical with θ'', σ', z' and $m \to \infty$.

It was shown in [4] that when $\theta'' \leqslant 4$ the optimum value of the dimensionless damping decrement $n = [\theta''(1+\theta'')^{-1}]^{1/2}/2$, and when $\theta'' > 4$, n is the minimum positive root of the polynomial $(1 + \theta'')^2n_6 + 3(1 + \theta'')n^4 - 3(\theta'' - 1)n^2 + 1 = 0$ and can be calculated using Cardano's formula. When $\theta'' \leqslant 4$ the dimensional damping decrement $\varepsilon = \Omega_1 n = \{mC[M_1(M_1 + m)]^{-1}\}^{1/2}/2$ is independent of M_2 if this quantity is sufficiently small, while if

$$mM_1^{-1} \leqslant 4 \tag{4}$$

the decrement ε is constant for all M_2. If m_1, m and C are constant and condition (4) is satisfied, then when $M_2 \to \infty$ the optimum values of the damper parameters will have the limits

$$\sigma' = \left(\frac{M_1}{M_1+m}\right)^2, \quad z' = 2M_1\left[\frac{m}{(M_1+m)^3}\right]^{1/2}, \quad c = \frac{CmM_1}{(M_1+m)^2}, \quad h = 2\left[CM_1\left(\frac{m}{M_1+m}\right)^3\right]^{1/2}$$

For the same M_2 for which $\theta'' > 4$, ε decreases as M_2 increases.

Consider the case of constant M_1, M_2 and C. Then the quantity Ω_1 will also be constant and for the same m for which $\theta'' < 4$ (and only for these), and n and ε will be increasing functions of m. If $M_2M_1^{-1} \leqslant 4$, then for all m we will have $\theta'' < 4$ and as $m \to \infty$ the limits of the optimum values of the damping decrement and the damper parameters will be as follows:

$$n = \frac{1}{2}\left(\frac{M_2}{M}\right)^{1/2}, \quad \varepsilon = \frac{1}{2}\left(\frac{C}{M_1}\right)^{1/2}, \quad \sigma' = \left(\frac{M_1}{M}\right)^2,$$

$$z' = 2M_1\left(\frac{M_2}{M^3}\right)^{1/2}, \quad c = \frac{CM_1}{M_2}, \quad h = 2(CM_1)^{1/2}$$

i.e. ε is half the natural frequency of the mass M, if it had been connected by a spring of stiffness C to a fixed foundation. If the masses M_1 and M_2 had been connected to it by springs of stiffness c and C, respectively, the natural frequencies of these masses would have been the same.

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Translated by R.C.G.